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About closed subsets of spaces of first category

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ABSTRACT

The space $h(X,k)$ is the smallest h-homogeneous space of first category and of weight k that contains a space X as a closed subset. We prove that if Y is a metric space of first category such that every nonempty open subset of Y contains a closed copy of X and has weight $\geqslant k$, then Y contains a closed copy of $h(X,k)$. This allows us to give an internal characterization of $h(X,k)$. We also establish some relations between homogeneous and h-homogeneous spaces.

1. Introduction

All spaces under discussion are metrizable.

Ostrovsky [1] and van Engelen [2] obtained (independently) that every u-homogeneous space of first category is h-homogeneous. We modify their technique to investigate closed subsets of metric spaces of first category. We prove (see Theorem 2) that if Y is a metric space of first category such that every nonempty open subset of Y contains a closed copy of X and has weight $\geqslant k$, then Y contains a closed copy of $h(X,k)$. The space $h(X,k)$ is called [3] the h-homogeneous extension of weight k of a space X with respect to spaces of first category. Briefly, $h(X,k)$ is the h-homogeneous extension of X. Of course, we define the space $h(X,k)$ only for X with $\operatorname{Ind} X = 0$. For each X of weight $\leqslant k$ the space $h(X,k)$ is unique (up to homeomorphism).

It is well known that the space Q of the rational numbers is a countable metric space without isolated points. Conversely, Sierpiński [4] proved that each countable metric space without isolated points is homeomorphic to Q. Similarly, according to [3, Theorem 6], the space $Y = h(X,k)$ has the following properties:

(a1) Y is of first category and $\operatorname{Ind} Y = 0$,
(a2) Y is a weight-homogeneous space of weight k,
(a3) every nonempty clopen subset of Y contains a closed copy of X,
(a4) $Y \in \sigma LF(X)$.

We shall prove (see Theorem 3) that each space Y satisfying the conditions (a1)–(a4) is homeomorphic to $h(X,k)$. The Sierpiński theorem is obtained when $k = \omega$ and X is a one-point set, because $Q \approx h(\{point\}, \omega)$.

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Theorem 4 shows how to convert a homogeneous space X with $\operatorname{Ind} X = 0$ into an h-homogeneous space. For this it suffices to multiply X by $Q(k)$, where $k = w(X)$. Theorem 5 states that a homogeneous, weight-homogeneous space X with $\operatorname{Ind} X = 0$ is h-homogeneous if the weight of X has uncountable cofinality.

2. Notation

For all undefined terms and notation see [5]. $X \approx Y$ means that X and Y are homeomorphic spaces. Let $\mathcal{P}$ be a topological property. Then a space X is *nowhere* $\mathcal{P}$ if no nonempty open subset of X has property $\mathcal{P}$.

We identify cardinals with initial ordinals; in particular, $\omega = \{0, 1, 2, \ldots\}$. We assume that all metrics are bounded by 1. A space is called *weight-homogeneous* if all nonempty open subspaces have the same weight.

A set $A \subseteq X$ is called *of first category* (or *meager*) in X if it is a countable union of nowhere dense subsets of X. A space X is of first category if it is of first category in itself. Clearly, every space of first category can be expressed as a countable union of its nowhere dense closed subsets.

A *clopen* set is a set which is both closed and open. A space X is *homogeneous* if for all $x, y \in X$, there exists a homeomorphism $f : X \to X$ with $f(x) = y$. A space X is called *h-homogeneous* if every nonempty clopen subset of X is homeomorphic to X and $\operatorname{Ind} X = 0$. Each h-homogeneous space is homogeneous, but the converse is false. For example, if X is a homogeneous space of weight k and locally of weight $<k$, then X is not h-homogeneous because every h-homogeneous space is weight-homogeneous. A space Y is called *u-homogeneous with respect to a space* X if every nonempty open subset of Y contains a closed (in Y) subset that is homeomorphic to X; in symbols, $Y \in u(X)$. Notice that Y may be a non zero-dimensional space in the last definition. We say that X is a *u-homogeneous* space if $X \in u(X)$ and $\operatorname{Ind} X = 0$. Clearly, each h-homogeneous space is u-homogeneous and each u-homogeneous space is weight-homogeneous.

For a family $\mathscr{U}$ of subsets of X, let $\bigcup \mathscr{U} = \bigcup\{U\colon\ U \in \mathscr{U}\}$ and $\operatorname{mesh} \mathscr{U} = \sup\{\operatorname{diam}(U)\colon\ U \in \mathscr{U}\}$. Next, if $f : X \to Y$ is a mapping, then we write $f(\mathscr{U}) = \{f(U)\colon\ U \in \mathscr{U}\}$.

For a space X define a family $LF(X) = \{Y\colon$ each point $y \in Y$ lies in some clopen neighborhood that is homeomorphic to a closed subset of $X\}$. A space $Y \in \sigma LF(X)$ if $Y = \bigcup\{Y_i\colon\ i \in \omega\}$, where each $Y_i \in LF(X)$ and each Y_i is a closed subset of Y.

Denote by $Q(k)$ the space $\{(x_0, x_1, \ldots) \in k^\omega\colon\ \exists i\ \forall j (j \geqslant i \Rightarrow x_j = 0)\}$. In [6] we proved that if X is a σ-discrete weight-homogeneous metric space of weight k, then $X \approx Q(k)$. Clearly, $Q(\omega) \approx Q$. So, we can consider $Q(k)$ as a nonseparable analogue of weight k for the space Q.

3. Main results

Lemma 1. *Let F be a nowhere dense closed subset of a metric space (X, ϱ) and $\operatorname{Ind} F = 0$. Let $\mathscr{W}_0, \mathscr{W}_1, \ldots$ be a sequence of discrete clopen (in F) covers of F such that $\operatorname{mesh}(\mathscr{W}_i) < (i + 1)^{-1}$ and $\mathscr{W}_{i+1}$ is a refinement of $\mathscr{W}_i$, where $i \in \omega$. Then there exist sequences of discrete open (in X) families $\mathscr{U}_0, \mathscr{U}_1, \ldots$ and $\mathscr{V}_0, \mathscr{V}_1, \ldots$ satisfying the following conditions for each $i \in \omega$:*

(b1) *$\mathscr{W}_i = \{U \cap F\colon\ U \in \mathscr{U}_i\}$ and $\operatorname{mesh}(\mathscr{U}_i) < 3(i + 1)^{-1}$,*
(b2) *for each $U_1 \in \mathscr{U}_{i+1}$ there exists a $U \in \mathscr{U}_i$ such that $\overline{U_1} \subset U$,*
(b3) *for each $V \in \mathscr{V}_{i+1}$ there exists a $U \in \mathscr{U}_i$ such that $\overline{V} \subset U \setminus (\overline{\bigcup \mathscr{U}_{i+1}})$,*
(b4) *$\overline{\bigcup \mathscr{V}_{j+1}} \cap \overline{\bigcup \mathscr{U}_i} = \emptyset$ and $\overline{\bigcup \mathscr{V}_j} \cap \overline{\bigcup \mathscr{V}_i} = \emptyset$ whenever $j < i$,*
(b5) *$\overline{\bigcup\{\bigcup \mathscr{V}_j\colon\ j \geqslant i\}} = F \bigcup (\bigcup\{\overline{V}\colon\ V \in \mathscr{V}_j,\ j \geqslant i\})$,*
(b6) *$\bigcap\{\bigcup \mathscr{U}_n\colon\ n \in \omega\} = \bigcap\{\overline{\bigcup \mathscr{U}_n}\colon\ n \in \omega\} = F$,*
(b7) *$\bigcap\{\overline{\bigcup\{\bigcup \mathscr{V}_j\colon\ j \geqslant n\}}\colon\ n \in \omega\} = F$ and $F \bigcap (\bigcup\{\overline{\bigcup \mathscr{V}_n}\colon\ n \in \omega\}) = \emptyset$.*

The bar denotes the closure in X.

Proof. We shall construct the families $\mathscr{U}_i$ and $\mathscr{V}_i$ by induction. Let $\mathscr{U}_{-1} = \{X\}$ and $\mathscr{V}_0 = \{\emptyset\}$. Assume that the families $\mathscr{U}_j$ and $\mathscr{V}_{j+1}$ have been defined for $j < i$. Consider $U \in \mathscr{U}_{i-1}$. Since $F \cap U$ is nowhere dense in U, there exist nonempty open subsets O_U and V_U of U such that $(F \cap U) \subset O_U$, O_U lies in the $(i + 1)^{-1}$-ball about $F \cap U$, $\overline{O_U} \subset U$, and $\overline{V_U} \cap \overline{O_U} = \emptyset$. Put $\mathscr{V}_{i+1} = \{V_U\colon\ U \in \mathscr{U}_{i-1}\}$.

The family $\mathscr{W}_i$ is discrete in X. Since every metric space is collectionwise normal (see [5, Theorem 5.1.18]), there exists a discrete family $\{U_W\colon\ W \in \mathscr{W}_i\}$ of open subsets of X such that $W \subset U_W$ for every $W \in \mathscr{W}_i$. For each $W \in \mathscr{W}_i$ consider a (unique) set $U^* \in \mathscr{U}_{i-1}$ with $W \subset U^*$. Replacing U_W by $U_W \cap O_{U^*}$ if it is necessary, we can assume that $U_W \subset O_{U^*}$. Clearly, the family $\mathscr{U}_i = \{U_W\colon\ W \in \mathscr{W}_i\}$ is discrete in X. The conditions (b1) and (b3) imply that $F \subset \overline{\bigcup\{\bigcup \mathscr{V}_j\colon\ j \geqslant i\}}$ for each $i \in \omega$. One can verify that the conditions (b1)–(b5) hold. This completes the induction.

Since $F = \bigcup \mathscr{W}_i$ for each $i \in \omega$, from (b1) and (b2) it follows (b6). Using (b3)–(b6), we obtain (b7). □

Definition 1. Let F be a closed subset of a metric space (X, ϱ) and $\operatorname{Ind} F = 0$. We say that families $\mathscr{W}_0, \mathscr{W}_1, \ldots$ form a *d-base* $\mathscr{W}$ for F if $\mathscr{W}_i$ is a discrete clopen (in F) cover of F, $\operatorname{mesh}(\mathscr{W}_i) < (i + 1)^{-1}$, and $\mathscr{W}_{i+1}$ is a refinement of $\mathscr{W}_i$, where $i \in \omega$. Clearly, $\mathscr{W} = \{W\colon\ W \in \mathscr{W}_i,\ i \in \omega\}$ is a base for F.

Under the conditions of Lemma 1, the family $\mathcal{U} = \{U\colon\ U \in \mathcal{U}_i,\ i \in \omega\}$ is called *an external base* for a subspace F of a space X that *is generated by the d-base* $\mathcal{W}$. The family $\mathcal{V} = \{V\colon\ V \in \mathcal{V}_i,\ i \in \omega\}$ is called a *residual family that is associated with* $\mathcal{W}$. Clearly, $\mathcal{V}$ refines $\mathcal{U}$. Below we shall assume that the families $\mathcal{W}$, $\mathcal{U}$, and $\mathcal{V}$ always satisfy the conditions (b1)–(b7).

Briefly, Lemma 1 states that for each d-base $\mathcal{W}$ for $F \subset X$ there exist an external base $\mathcal{U}$ and a residual family $\mathcal{V}$ in X.

Let us consider an indexed family $\mathcal{W}_i = \{W_\alpha\colon\ \alpha \in A_i\}$, where $i \in \omega$. From the proof of Lemma 1 it follows that the families $\mathcal{U}_i$ and $\mathcal{V}_{i+1}$ can be indexed with the same set A_i. According to (b1), the set $U_\alpha \in \mathcal{U}_i$ is assigned to $W_\alpha \in \mathcal{W}_i$ providing $W_\alpha = U_\alpha \cap F$. By (b3), the set $V_\alpha \in \mathcal{V}_{i+1}$ corresponds to $W_\alpha \in \mathcal{W}_i$ if $V_\alpha \subset U_\alpha$. So, $\mathcal{U}_i = \{U_\alpha\colon\ \alpha \in A_i\}$ and $\mathcal{V}_{i+1} = \{V_\alpha\colon\ \alpha \in A_i\}$.

In fact, each family $\mathcal{U}_i$ is a swelling of the family $\mathcal{W}_i$ and the base $\mathcal{U}$ is a swelling of the base $\mathcal{W}$. Note that for a given d-base $\mathcal{W}$ we can construct different external bases and residual families.

When $\operatorname{Ind} X = 0$, we can specify Lemma 1.

Lemma 2. *Let F be a nowhere dense closed subset of a metric space (X, ϱ) and* $\operatorname{Ind} X = 0$. *Let $\mathcal{W}_0, \mathcal{W}_1, \ldots$ be a sequence of discrete clopen (in F) covers of F such that* $\operatorname{mesh}(\mathcal{W}_i) < (i+1)^{-1}$ *and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$, where $i \in \omega$. Then there exist sequences of discrete clopen (in X) families $\mathcal{U}_0, \mathcal{U}_1, \ldots$ and $\mathcal{V}_0, \mathcal{V}_1, \ldots$ satisfying the following conditions for each $i \in \omega$:*

(c1) $\mathcal{W}_i = \{U \cap F\colon\ U \in \mathcal{U}_i\}$ *and* $\operatorname{mesh}(\mathcal{U}_i) < 3(i+1)^{-1}$,
(c2) *for each $U_1 \in \mathcal{U}_{i+1}$ there exists a $U \in \mathcal{U}_i$ such that $U_1 \subset U$,*
(c3) *for each $V \in \mathcal{V}_{i+1}$ there exists a $U \in \mathcal{U}_i$ such that $V = U \setminus (\bigcup \mathcal{U}_{i+1})$,*
(c4) $\bigcup \mathcal{U}_i = \overline{\bigcup \mathcal{U}_i}$ *is a clopen neighborhood of F and* $\bigcup \mathcal{V}_i = \overline{\bigcup \mathcal{V}_i}$,
(c5) $\bigcup\{\bigcup \mathcal{V}_{j+1}\colon\ j \leqslant i\} = X \setminus (\bigcup \mathcal{U}_i)$,
(c6) $\overline{\bigcup\{\bigcup \mathcal{V}_j\colon\ j \geqslant i\}} = F \bigcup(\bigcup\{\bigcup \mathcal{V}_j\colon\ j \geqslant i\})$,
(c7) $\bigcap\{\bigcup \mathcal{U}_n\colon\ n \in \omega\} = F$,
(c8) $\bigcap\{\overline{\bigcup\{\bigcup \mathcal{V}_j\colon\ j \geqslant n\}}\colon\ n \in \omega\} = F$ *and* $\bigcup\{\bigcup \mathcal{V}_n\colon\ n \in \omega\} = X \setminus F$.

The bar denotes the closure in X.

Proof. We shall construct the families $\mathcal{U}_i$ and $\mathcal{V}_i$ by induction. Let $\mathcal{U}_{-1} = \{X\}$ and $\mathcal{V}_0 = \{\emptyset\}$. Suppose that the families $\mathcal{U}_j$ and $\mathcal{V}_{j+1}$ have been obtained for $j < i$. Take $U \in \mathcal{U}_{i-1}$. Since F is nowhere dense in X, there exist nonempty clopen subsets O_U and V_U of U such that $(F \cap U) \subset O_U$, O_U is contained in the $(i+1)^{-1}$-ball about $F \cap U$, and $V_U = U \setminus O_U$. Define $\mathcal{V}_{i+1} = \{V_U\colon\ U \in \mathcal{U}_{i-1}\}$. Fix a retraction $r_U : O_U \to O_U$ with $r_U(O_U) = F \cap U$. Put $\mathcal{W}_U = \{r_U^{-1}(W)\colon\ W \subset U \text{ and } W \in \mathcal{W}_i\}$. Define the family $\mathcal{U}_i = \{U^*\colon\ U^* \in \mathcal{W}_U,\ U \in \mathcal{U}_{i-1}\}$. This completes the induction.

As in the proof of Lemma 1 one can verify that the conditions (c1)–(c8) hold for families $\mathcal{U}_i$ and $\mathcal{V}_i$. $\square$

Definition 2. Let $\mathcal{W}_j$ be a d-base for a nowhere dense closed subset F_j of a space X_j and $\mathcal{V}_j$ be a residual family that is associated with $\mathcal{W}_j$ in X_j, where $j \in \{1, 2\}$. For a given homeomorphism $f : F_1 \to F_2$ we say that a bijection $\psi : \mathcal{V}_1 \to \mathcal{V}_2$ *is agreed to* f if for any subsets $D_1 \subseteq \bigcup \mathcal{V}_1$ and $D_2 \subseteq \bigcup \mathcal{V}_2$ and any bijection $g : D_1 \to D_2$ satisfying $g(D_1 \cap V) = \psi(V) \cap D_2$ for every $V \in \mathcal{V}_1$, the combination mapping $f \nabla g : F_1 \cup D_1 \to F_2 \cup D_2$ is continuous at each point of F_1 and the inverse mapping $(f \nabla g)^{-1}$ is continuous at each point of F_2.

By the definition of a residual family, $F_1 \cap (\bigcup \mathcal{V}_1) = \emptyset$ and $F_2 \cap (\bigcup \mathcal{V}_2) = \emptyset$. Hence, the mapping $f \nabla g : F_1 \cup D_1 \to F_2 \cup D_2$ is well defined.

Remark 1. For a zero-dimensional space X a similar construction was applied by Ostrovsky [1], van Engelen [2], and the author [7]. According to van Engelen, the families $\mathcal{V}_1$ and $\mathcal{V}_2$ from Lemma 2 form *a KR-covering for* $(X_1 \setminus F_1, X_2 \setminus F_2, f)$. Notice that in Definition 2 and Lemma 3 it may be $\operatorname{Ind} X \neq 0$. Hence, we do not assume that $\bigcup \mathcal{V}_i = X_i \setminus F_i$ for $i \in \{1, 2\}$.

Lemma 3. *Let F_j be a nowhere dense closed subset of a metric space (X_j, ϱ_j),* $\operatorname{Ind} F_j = 0$, *where $j \in \{1, 2\}$. Let $f : F_1 \to F_2$ be a homeomorphism. Then there exists a d-base $\mathcal{W}_1 = \{\mathcal{W}_i^1\colon\ i \in \omega\}$ for F_1 in X_1 and there exists a d-base $\mathcal{W}_2 = \{\mathcal{W}_i^2\colon\ i \in \omega\}$ for F_2 in X_2 satisfying the following conditions:*

(d1) $f(\mathcal{W}_i^1) = \mathcal{W}_i^2$ *for every $i \in \omega$,*
(d2) *for any residual family $\mathcal{V}_1$ that is associated with $\mathcal{W}_1$ in the space X_1 and any residual family $\mathcal{V}_2$ that is associated with $\mathcal{W}_2$ in the space X_2 there is a bijection $\psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that ψ is agreed to f.*

Proof. The sequences $\mathcal{W}_0^1, \mathcal{W}_1^1, \ldots$ and $\mathcal{W}_0^2, \mathcal{W}_1^2, \ldots$ will be defined by induction. Assume that $n = 0$ or that $n > 0$ and the covers $\mathcal{W}_i^j$ of F_j are already defined for $i < n$ and $j \in \{1, 2\}$. For every $x \in F_1$ take a clopen (in F_1) neighborhood U_x of the point x such that $\operatorname{diam}(U_x) < (n+1)^{-1}$ and U_x is contained in a member of $\mathcal{W}_{n-1}^j$, if $n > 0$. By virtue of the Dowker theorem (see [5, Theorem 7.2.4]), the cover $\{U_x\colon x \in F_1\}$ of F_1 has a discrete clopen refinement $\mathscr{W}_n^{*1}$. Then $f(\mathscr{W}_n^{*1})$ is a discrete clopen (in F_2) cover of F_2. As above, for every $y \in F_2$ take a clopen (in F_2) neighborhood U_y of the point y such that $\operatorname{diam}(U_y) < (n+1)^{-1}$ and U_y is contained in a member of $f(\mathscr{W}_n^{*1})$. The cover $\{U_y\colon y \in F_2\}$ of F_2 has a discrete clopen refinement $\mathscr{W}_n^2$. Put $\mathscr{W}_n^1 = f^{-1}(\mathscr{W}_n^2)$. This completes the induction. Clearly, the sequence $\mathscr{W}_0^j, \mathscr{W}_1^j, \ldots$ forms a d-base $\mathscr{W}_j$ for F_j, where $j \in \{1,2\}$. By construction, the condition (d1) holds.

For $j \in \{1,2\}$ consider a residual family $\mathscr{V}_j$ that is associated with $\mathscr{W}_j$ in X_j. According to Definition 1, take an external base $\mathscr{U}_j = \{\mathscr{U}_i^j\colon i \in \omega\}$ for F_j in X_j such that $\mathscr{U}_j$ is generated by the d-base $\mathscr{W}_j$ and $\mathscr{V}_j$ refines $\mathscr{U}_j$. Suppose that for each $i \in \omega$ the family $\mathscr{W}_i^1$ is indexed by a set A_i, i.e., $\mathscr{W}_i^1 = \{W_{i,\alpha}^1\colon \alpha \in A_i\}$. From (d1) it follows that $\mathscr{W}_i^2 = \{W_{i,\alpha}^2\colon \alpha \in A_i\}$ if we put $W_{i,\alpha}^2 = f_0(W_{i,\alpha}^1)$. Then $\mathscr{U}_i^j = \{U_{i,\alpha}^j\colon \alpha \in A_i\}$ and $\mathscr{V}_{i+1}^j = \{V_{i+1,\alpha}^j\colon \alpha \in A_i\}$ for each $i \in \omega$. Hence, $\mathscr{V}_j = \{V_{i+1,\alpha}^j\colon \alpha \in A_i,\ i \in \omega\}$. Define the bijection $\psi : \mathscr{V}_1 \to \mathscr{V}_2$ by the rule $\psi(V_{i+1,\alpha}^1) = V_{i+1,\alpha}^2$. Since $\mathscr{V}_0^1 = \mathscr{V}_0^2 = \{\emptyset\}$ by definition, for $V \in \mathscr{V}_0^1$ let $\psi(V) = \{\emptyset\}$.

Take subsets $D_1 \subseteq \bigcup \mathscr{V}_1$ and $D_2 \subseteq \bigcup \mathscr{V}_2$ and any bijection $g : D_1 \to D_2$ such that $g(D_1 \cap V) = \psi(V) \cap D_2$ for every $V \in \mathscr{V}_1$.

Consider a point $x \in F_1$ and a neighborhood O of the point $y = f(x) \in F_2$. By (b1) and (b6) there exist $i \in \omega$ and $\beta \in A_i$ such that $y \in U_{i,\beta}^2 \subseteq O$. From (b1) and (d1) it follows that $f^{-1}(F_2 \cap U_{i,\beta}^2) = F_1 \cap U_{i,\beta}^1$. By construction of a residual family, $g^{-1}(E_2 \cap U_{i,\beta}^2) = E_1 \cap U_{i,\beta}^1$. The set $(F_1 \cup E_1) \cap U_{i,\beta}^1$ is open in $F_1 \cup E_1$ and contains the point x. Hence, $f \nabla g$ is continuous at the point $x \in F_1$. Similarly, $(f \nabla g)^{-1}$ is continuous at any point $y \in F_2$. $\square$

Lemma 4. *Let* $\operatorname{Ind} X = 0$ *and* $X = \bigcup\{X_i\colon i \in \omega\}$*, where each* X_i *is a nowhere dense closed subset of* X*. Let* Y *be a space of first category and* Y *be a u-homogeneous space with respect to each* X_i*. Then* Y *contains a nowhere dense closed copy of* X*.*

Furthermore, every nonempty open subset of Y *contains a nowhere dense closed copy of* X*.*

Proof. Since $\operatorname{Ind} X = 0$, we can assume that $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Let $Y = \bigcup\{Y_i\colon i \in \omega\}$, where each Y_i is a nowhere dense closed subset of Y.

Put $F_0 = X_0$. Choose a homeomorphic embedding $f_0 : F_0 \to Y$ such that the image $Z_0 = f_0(F_0)$ is a nowhere dense closed subset of Y. By Lemma 3, there exist a d-base $\mathscr{W} = \{\mathscr{W}_i\colon i \in \omega\}$ for a subset F_0 of X and the d-base $\mathscr{Z} = \{\mathscr{Z}_i\colon i \in \omega\}$ for a subset Z_0 of Y such that $f_0(\mathscr{W}_i) = \mathscr{Z}_i$ for every $i \in \omega$. Since $\operatorname{Ind} X = 0$, by Lemma 2 take a residual family $\mathscr{V}_0$ such that $\mathscr{V}_0$ is associated with $\mathscr{W}$ and $\bigcup \mathscr{V}_0 = X \setminus F_0$. Note that $\mathscr{V}_0$ consists of nonempty clopen subsets of X. Using Lemma 1, take a residual family $\mathscr{C}_0$ that is associated with $\mathscr{Z}$ in Y. From Lemma 3 it follows that there exists the bijection $\psi_0 : \mathscr{V}_0 \to \mathscr{C}_0$ that is agreed to f_0. Since Y_0 is a nowhere dense closed subset of the space Y of first category, for every $C \in \mathscr{C}_0$ there exists a nonempty open subset $D_C \subset C$ such that $\overline{D_C} \cap Y_0 = \emptyset$, $\overline{D_C} \subset C$, and the interior of the set $C \setminus D_C$ is nonempty. Let $\mathscr{D}_0 = \{D_C\colon C \in \mathscr{C}_0\}$. By construction, $\bigcup \mathscr{D}_0 \subset \bigcup \mathscr{C}_0$, $\overline{\bigcup \mathscr{D}_0} = Z_0 \bigcup(\bigcup\{\overline{D}\colon D \in \mathscr{D}_0\})$, and $(\bigcup\{\overline{D}\colon D \in \mathscr{D}_0\}) \cap (Z_0 \cup Y_0) = \emptyset$.

Next, consider a $V \in \mathscr{V}_0$, $C = \psi_0(V) \in \mathscr{C}_0$, and $D_C \subset C$. Find the smallest number j such that the intersection $X_V = V \cap X_j$ is nonempty. Choose a homeomorphic embedding $f_{1,V} : X_V \to \psi_0(V)$ such that the image $Z_V = f_{1,V}(X_V)$ is a nowhere dense closed subset of D_C. As above, take a residual family $\mathscr{V}_V$ such that $\mathscr{V}_V$ is associated with $\mathscr{W}_V$ for some d-base $\mathscr{W}_V$ for X_V in V and $\bigcup \mathscr{V}_V = V \setminus X_V$. Choose a residual family $\mathscr{C}_V$ such that $\bigcup \mathscr{C}_V \subseteq D_C \setminus Z_V$ and $\mathscr{C}_V$ is associated with the d-base $\mathscr{Z}_V = f_{1,V}(\mathscr{W}_V)$ for Z_V in D_C. Without loss of generality, we can assume that $\operatorname{mesh}(\mathscr{V}_V) < 2^{-1}$ and $\operatorname{mesh}(\mathscr{C}_V) < 2^{-1}$. By Lemma 3, there exists the bijection $\psi_{1,V} : \mathscr{V}_V \to \mathscr{C}_V$ that is agreed to $f_{1,V}$. Since Y_1 is a nowhere dense closed subset of Y, for every $C^* \in \mathscr{C}_V$ there exists a nonempty open subset $D_{C^*} \subset C^*$ such that $\overline{D_{C^*}} \cap Y_1 = \emptyset$, $\overline{D_{C^*}} \subset C^*$, and $\operatorname{Int}(C^* \setminus D_{C^*}) \neq \emptyset$. Let $\mathscr{D}_V = \{D_{C^*}\colon C^* \in \mathscr{C}_V\}$.

Define $F_1 = F_0 \cup (\bigcup\{X_V\colon V \in \mathscr{V}_0\})$ and $Z_1 = Z_0 \cup (\bigcup\{Z_V\colon V \in \mathscr{V}_0\})$. From (b3) and (b4) it follows that F_1 is a nowhere dense closed subset of X and Z_1 is a nowhere dense closed subset of Y. By construction, $X_1 \subseteq F_1$. Define the mapping $f_1 : F_1 \to Z_1$ by the rule $f_1(x) = f_0(x)$ if $x \in F_0$; or $f_1(x) = f_{1,V}(x)$ if $x \in X_V$ for a $V \in \mathscr{V}_0$. Clearly, f_1 is a bijection. For every $V \in \mathscr{V}_0$ the restriction $f_1|X_V$ is a homeomorphism. From Lemma 3 it follows that f_1 is a homeomorphism.

Let $\mathscr{V}_1 = \{V^*\colon V^* \in \mathscr{V}_V,\ V \in \mathscr{V}_0\}$, $\mathscr{C}_1 = \{C^*\colon C^* \in \mathscr{C}_V,\ V \in \mathscr{V}_0\}$, and $\mathscr{D}_1 = \{D^*\colon D^* \in \mathscr{D}_V,\ V \in \mathscr{V}_0\}$. By construction, $\bigcup \mathscr{V}_1 = X \setminus F_1$. One can check that $(\bigcup \mathscr{D}_1) \subset (\bigcup \mathscr{C}_1) \subset (\bigcup \mathscr{D}_0)$ and $(\bigcup\{\overline{D}\colon D \in \mathscr{D}_1\}) \cap (Z_1 \cup Y_0 \cup Y_1) = \emptyset$. The property (b7) implies that the closure of $\bigcup \mathscr{D}_1$ coincides with $Z_1 \cup (\bigcup\{\overline{D}\colon D \in \mathscr{D}_1\})$. The family $\mathscr{V}_1$ consists of pairwise disjoint clopen subsets of X. The family $\mathscr{C}_1$ consists of open subsets of Y with pairwise disjoint closures and the family $\mathscr{D}_1$ is the same. Define the bijection $\psi_1 : \mathscr{V}_1 \to \mathscr{C}_1$ by the rule $\psi_1(V^*) = \psi_{1,V}(V^*)$ providing $V^* \in \mathscr{V}_V$ for a $V \in \mathscr{V}_0$. Clearly, $\operatorname{mesh}(\mathscr{V}_1) < 2^{-1}$ and $\operatorname{mesh}(\mathscr{C}_1) < 2^{-1}$.

Next, fix a $V \in \mathscr{V}_1$ and so on. As a result, by induction, for each $n \in \omega$ we can construct nowhere dense closed subsets $F_n \subset X$ and $Z_n \subset Y$, a homeomorphism $f_n : F_n \to Z_n$, a family $\mathscr{V}_n$ consisting of pairwise disjoint clopen subsets of X, a family $\mathscr{C}_n$ consisting of open subsets of Y with pairwise disjoint closures, a family $\mathscr{D}_n = \{D_C\colon C \in \mathscr{C}_n\}$, and a bijection $\psi_n : \mathscr{V}_n \to \mathscr{C}_n$ satisfying the following conditions:

(e1) $F_0 = X_0$, $X_{n+1} \cup F_n \subset F_{n+1}$, and $\bigcup \mathscr{V}_n = X \setminus F_n$,
(e2) for every $V^* \in \mathscr{V}_{n+1}$ there exists a $V \in \mathscr{V}_n$ such that $V^* \subset V$,
(e3) $\overline{D_{C^*}} \subset C^*$ and $\operatorname{Int}(C^* \setminus D_{C^*}) \neq \emptyset$ for every $C \in \mathscr{C}_n$,
(e4) $Z_n \subset Z_{n+1}$, $(Z_{n+1} \setminus Z_n) \subset \bigcup \mathscr{D}_n \subset \bigcup \mathscr{C}_n$, and $\bigcup \mathscr{C}_{n+1} \subset \bigcup \mathscr{D}_n$,

(e5) $\overline{\bigcup \mathscr{D}_n} = Z_n \cup (\bigcup\{\overline{D}\colon D \in \mathscr{D}_n\})$ and $\overline{\bigcup \mathscr{C}_n} = Z_n \cup (\bigcup\{\overline{C}\colon C \in \mathscr{C}_n\})$,
(e6) $(\bigcup\{\overline{D}\colon D \in \mathscr{D}_n\}) \cap (\bigcup\{Y_i\colon i \leqslant n\}) = \emptyset$ and $Z_n \cap (\bigcup\{\overline{D}\colon D \in \mathscr{D}_n\}) = \emptyset$,
(e7) $\operatorname{mesh}(\mathscr{V}_n) < (n+1)^{-1}$ and $\operatorname{mesh}(\mathscr{C}_n) < (n+1)^{-1}$,
(e8) $V \cap F_{n+1} \neq \emptyset$ and $f_{n+1}(V \cap F_{n+1}) = D_C \cap Z_{n+1}$, where $C = \psi_n(V)$, for every $V \in \mathscr{V}_n$,
(e9) the restriction of f_{n+1} to F_n coincides with f_n.

From (e1) it follows that $X = \bigcup\{F_n\colon n \in \omega\}$. Let $Z^* = \bigcup\{Z_n\colon n \in \omega\}$. Define the mapping $f : X \to Z^*$ by the rule $f(x) = f_n(x)$ if $x \in F_n$. According to (e9) and (e1), f is well defined. Clearly, f is a bijection. Lemma 3 implies that f is continuous at each point of F_n and f^{-1} is continuous at each point of Z_n for every $n \in \omega$. Hence, f is a homeomorphism.

Let us verify that Z^* is a closed subset of Y. Assume that there exists a point $y \in \overline{Z^*} \setminus Z^*$. Take the smallest n such that $y \in Y_n$. By (e6), $y \notin \bigcup\{\overline{D}\colon D \in \mathscr{D}_n\}$. On the other hand, consider the set $B_n = \bigcup\{Z_i\colon i > n\}$. From (e4) it follows that $B_n \subset \bigcup \mathscr{D}_n$. Since $Z^* = Z_n \cup B_n$ and Z_n is closed in Y, (e5) implies that

$$\overline{Z^*} \subseteq Z_n \cup \overline{B_n} \subseteq Z_n \cup \overline{\left(\bigcup \mathscr{D}_n\right)} = Z_n \cup \left(\bigcup\{\overline{D}\colon D \in \mathscr{D}_n\}\right).$$

Hence, the point $y \in Z^* \cup (\bigcup\{\overline{D}\colon D \in \mathscr{D}_n\})$. A contradiction. So, $\overline{Z^*} = Z^*$.

It remains to check that Z^* is nowhere dense in Y. Consider an open subset O of Y such that $O \cap Z^* \neq \emptyset$. Fix a point $y \in O \cap Z^*$ and choose the smallest n with $y \in Z_n$. By construction, there exists an open subset O^* of Y such that $y \in O^*$ and the family $\mathscr{C}^* = \{C\colon C \subset O^*,\ C \in \mathscr{C}_n\}$ forms a residual family that is associated with some d-base $\mathscr{Z}_O = \{\mathscr{Z}_i^O\colon i \in \omega\}$, where $\bigcup \mathscr{Z}_O = O^* \cap Z_n$. Let $\mathscr{U}_O$ be an external base that corresponds to $\mathscr{C}^*$. Choose a $j \in \omega$ such that the $(3j+3)^{-1}$-ball about y lies in $O \cap O^*$. From the properties (b3) and (b1) of the external base $\mathscr{U}_O$ it follows that there exists an open subset $C^* \in \mathscr{C}^*$ of Y such that the $(3j+3)^{-1}$-ball about y contains C^*. According to (e3), $\operatorname{Int}(C^* \setminus D_{C^*}) \neq \emptyset$. From (e4) and (e6) it follows that $Z^* \cap \operatorname{Int}(C^* \setminus D_{C^*}) = \emptyset$. Hence, Z^* is nowhere dense in O.

Consider a nonempty open subset U of Y. Clearly, the conditions of the theorem hold with respect to U. Hence, U contains a nowhere dense closed copy of X. $\square$

Theorem 1. *Suppose that* $\operatorname{Ind} X = 0$*, Y is of first category, Y is nowhere locally of weight $< k$, and Y is u-homogeneous with respect to X. Let $Z \in \sigma LF(X)$ and $w(Z) \leqslant k$. Then Y contains a nowhere dense closed copy of Z.*

Proof. By definition, $Z = \bigcup\{Z_i\colon i \in \omega\}$, where each Z_i is closed in Z and $Z_i \in LF(X)$. From the locally finite sum theorem (see [5]) it follows that $\operatorname{Ind} Z = 0$.

Let us verify that Y is u-homogeneous with respect to each Z_i. Fix $i \in \omega$. Take a nonempty open subset U of Y. By virtue of the Stone theorem [5, Exercise 5.3.A], we have $Z_i = \bigoplus\{Z_{i,\alpha}\colon \alpha \in A_i\}$, where every $Z_{i,\alpha}$ is homeomorphic to a closed subset of X and $|A_i| \leqslant k$. Since Y is nowhere locally of weight $<k$, there exists a discrete open family $\mathscr{U}_i = \{U_{i,\alpha}\colon \alpha \in A_i\}$ with $\bigcup \mathscr{U}_i \subset U$. Under the conditions of the theorem, every $U_{i,\alpha}$ is of first category and contains a closed copy of X. By Lemma 4, every $U_{i,\alpha}$ contains a closed copy of $Q \times X$. In particular, $U_{i,\alpha}$ contains a nowhere dense closed copy of $Z_{i,\alpha}$. Hence, U contains a nowhere dense closed copy of Z_i.

Lemma 4 implies that Y contains a nowhere dense closed copy of Z. $\square$

Corollary 1. *Let* $\operatorname{Ind} X = 0$*, $Z \in \sigma LF(X)$, and $w(Z) \leqslant k$. Then $h(X,k)$ contains a nowhere dense closed copy of Z.*

Proof. The space $Y = h(X,k)$ satisfies the conditions (a1)–(a4) from Introduction. It remains to apply Theorem 1. $\square$

Theorem 2. *Suppose that* $\operatorname{Ind} X = 0$*, every nonempty open subset of Y has weight $\geqslant k$ and contains a closed copy of X, and Y is of first category. Then Y contains a nowhere dense closed copy of $h(X,k)$.*

Proof. Since the h-homogeneous extension $h(X,k) \in \sigma LF(X)$ and weight of $h(X,k)$ is equal to k, the theorem follows from Theorem 1. $\square$

Corollary 2. *([8]) Every space of first category, which is nowhere locally of weight $<k$, contains a nowhere dense closed copy of $Q\,(k)$.*

Proof. The corollary follows from Theorem 2 when $X = \{point\}$. $\square$

Theorem 3. *Suppose that* $\operatorname{Ind} X = 0$*, every nonempty open subset of Y has weight k and contains a closed copy of X, Y is of first category, and $Y \in \sigma LF(X)$. Then Y is homeomorphic to $h(X,k)$.*

Proof. From the locally finite sum theorem (see [5]) it follows that $\operatorname{Ind} Y = 0$. By Theorem 2, every nonempty open subset of Y contains a closed copy of $h(X,k)$. On the other hand, since $h(X,k)$ is an h-homogeneous space, Corollary 1 implies that every nonempty open subset of $h(X,k)$ contains a closed copy of Y. By virtue of the Ostrovsky theorem [1, Proposition B] (or by virtue of the van Engelen result [2, Lemma 3.1]), Y is homeomorphic to $h(X,k)$. $\square$

Theorem 4. *Let X be a homogeneous space of weight k and* $\operatorname{Ind} X = 0$. *Then $Q(k) \times X$ is homeomorphic to $h(X, k)$. In particular, $Q(k) \times X$ is h-homogeneous.*

Proof. Denote the product $Q(k) \times X$ by Y. Clearly, Y is a weight-homogeneous space of first category and $Y \in \sigma LF(X)$. According to Theorem 3, to prove the theorem it suffices to show that every nonempty open subset U of Y contains a closed copy of X.

By definition of the Tychonoff topology on Y, U contains a product $V \times W$, where V is a clopen subset of $Q(k)$ and W is a clopen subset of X. Take a discrete closed subset $D \subset V$ of cardinality k. Fix a point $a \in W$. Since X is a homogeneous space, for each $x \in X$ there exists a homeomorphism $f_x : X \to X$ with $f_x(a) = x$. Then $f_x(W)$ is a clopen neighborhood of x. By virtue of the Stone theorem [5, Theorem 4.4.1] and the Dowker theorem [5, Theorem 7.2.4], the cover $\{f_x(W): x \in X\}$ of X has a discrete open refinement $\{W_\alpha: \alpha \in A\}$ with $|A| \leqslant k$. Then each W_α is clopen in X. By construction, every W_α is homeomorphic to a clopen subset of W. From $|A| \leqslant k = |D|$ it follows that X is homeomorphic to a clopen subset of $D \times W$. So, U contains a closed copy of X. □

Corollary 3. *Let X be a separable homogeneous zero-dimensional space. Then $Q \times X$ is an h-homogeneous space.*

Theorem 5. *Let X be a homogeneous, weight-homogeneous space of weight k, k has uncountable cofinality, and* $\operatorname{Ind} X = 0$. *Then X is an h-homogeneous space.*

Proof. Let us show that X has a π-base consisting of clopen sets which are homeomorphic to X.

Fix a nonempty clopen subset $U \subseteq X$. Since X is weight-homogeneous and $k > \omega$, there is a discrete clopen cover $\mathscr{U}$ of U of cardinality k. Choose a point $a \in X$ and a decreasing base $\{V_n: n \in \omega\}$ at the point a consisting of clopen subsets of X. Denote by k_n the cardinality of the family $\mathscr{U}_n = \{U \in \mathscr{U}: U$ contains a clopen copy of $V_n\}$. Then $k_0 \leqslant k_1 \leqslant \ldots$. From homogeneity of X it follows that every $U \in \mathscr{U}$ contains a clopen copy of V_i for some $i \in \omega$. Hence, $\sup\{k_n: n \in \omega\} = k$. Since $\mathrm{cf}(k) > \omega$, we can find a j with $k_j = k$. Then the clopen subset $\bigcup \mathscr{U}_j$ of U contains a clopen subset F that is homeomorphic to $\bigoplus\{F_\alpha: \alpha \in k\}$, where each $F_\alpha \approx V_j$. Clearly, F is clopen in U.

Since X is a homogeneous space, for each $x \in X$ there exists a homeomorphism $f_x : X \to X$ with $f_x(a) = x$. Then $f_x(V_j)$ is a clopen neighborhood of x. By virtue of the Stone theorem [5, Theorem 4.4.1] and the Dowker theorem [5, Theorem 7.2.4], the cover $\{f_x(V_j): x \in X\}$ of X has a discrete open refinement $\{W_\alpha: \alpha \in A\}$ with $|A| \leqslant k$. Hence, each W_α is clopen in X. By construction, every W_α is homeomorphic to a clopen subset of $F_\alpha \approx V_j$. From $|A| \leqslant k$ it follows that X is homeomorphic to a clopen subset of F. So, U contains a clopen subset U^* which is homeomorphic to X.

If we take a clopen base $\mathcal{B}$ for X, then the family $\{U^*: U \in \mathcal{B}\}$ forms a π-base consisting of clopen sets which are homeomorphic to X. Every non-separable metric space is non-pseudocompact. From the Terada theorem [9, Theorem 2.4] it follows that X is an h-homogeneous space. □

In light of Corollary 3 and Theorem 5 the following question remains open.

Question 1. *Is there a separable zero-dimensional homogeneous space of first category which is not h-homogeneous?*

Remark 2. Let X be a homogeneous Borel subset of the Cantor set. Van Engelen [10] proved that if X is not locally compact, then X is h-homogeneous.

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